Solution and Parameter Estimation
in
Groundwater Model

by

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OUTLINE

1. Introduce Groundwater Model from [Sudicky et. al.] and [Drake et. al.] tracking contaminant dynamics in Groundwater flowing through fissures (cracks) in rock matrix via schematic diagram.

2. Present coupled PDE (Partial Differential Equation) representing model.

3. Describe terms and parameters in PDE model.

4. Use implicit finite difference scheme to numerically estimate model solution.

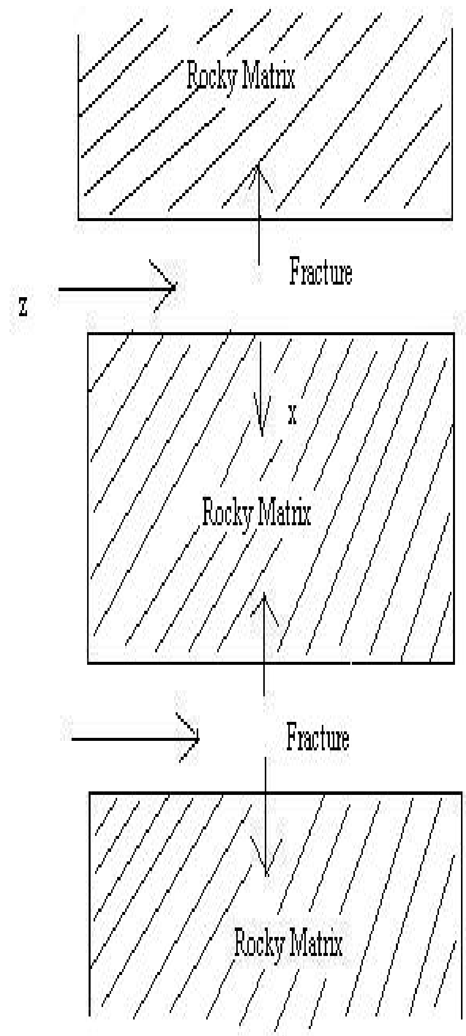
5. Briefly discuss convergence of finite difference scheme and present graphical examples.

6. Discuss inverse method procedure to numerically approximate model parameters.

7. Briefly discuss convergence results for parameter approximants and present graphical illustrations.

8. Mention future Research.

SCHEMATIC DIAGRAM



PDE MODEL

$$\begin{cases} \dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial z} = b\dfrac{\partial^2 u}{\partial z^2} - \lambda u \\ \dfrac{\partial v}{\partial t} = e\dfrac{\partial^2 v}{\partial x^2} - \lambda v \end{cases}$$

Equation 1

with initial and boundary conditions

$$\begin{cases} u(0,\ z) = \alpha(z) \\ u(t,\ 0) = \gamma(t) \\ v(0,\ x,\ z) = \eta(x,\ z) \\ v(t,\ 0,\ z) = \rho\ u(t,\ z) \\ v(t,\ x,\ 0) = \rho\ e^{-x}\ \gamma(t) \end{cases}$$

Equation 2

FINITE DIFFERENCE SCHEME

$$\begin{cases} \dfrac{u_l^{k+1} - u_l^k}{\Delta t} + a\dfrac{u_{l+1}^{k+1} - u_{l-1}^{k+1}}{2\Delta z} = b\dfrac{u_{l+1}^{k+1} - 2u_l^k + u_{l-1}^{k+1}}{(\Delta z)^2} - \lambda u_l^k \\ \\ \dfrac{v_{j,l}^{k+1} - v_{j,l}^k}{\Delta t} = e\dfrac{v_{j+1,l}^{k+1} - 2v_{j,l}^{k+1} + v_{j-1,l}^{k+1}}{(\Delta x)^2} - \lambda v_{j,l}^k \end{cases}$$

Equation 3

Fact: $\{(u_l^k, v_{j,l}^k)\} \to (u, v)$ in $\|\cdot\|_\infty$ with quadratic order of convergence.

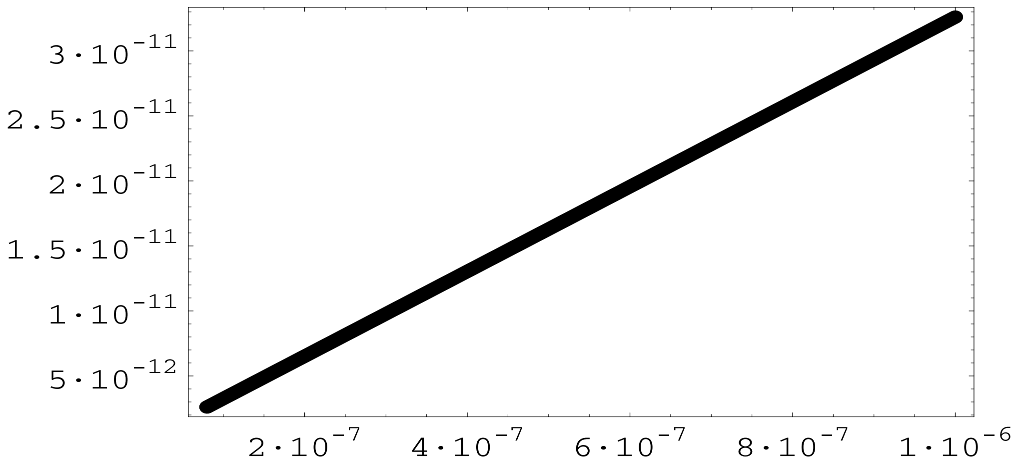


Figure 1: Square of Step Size vs Relative Error of u

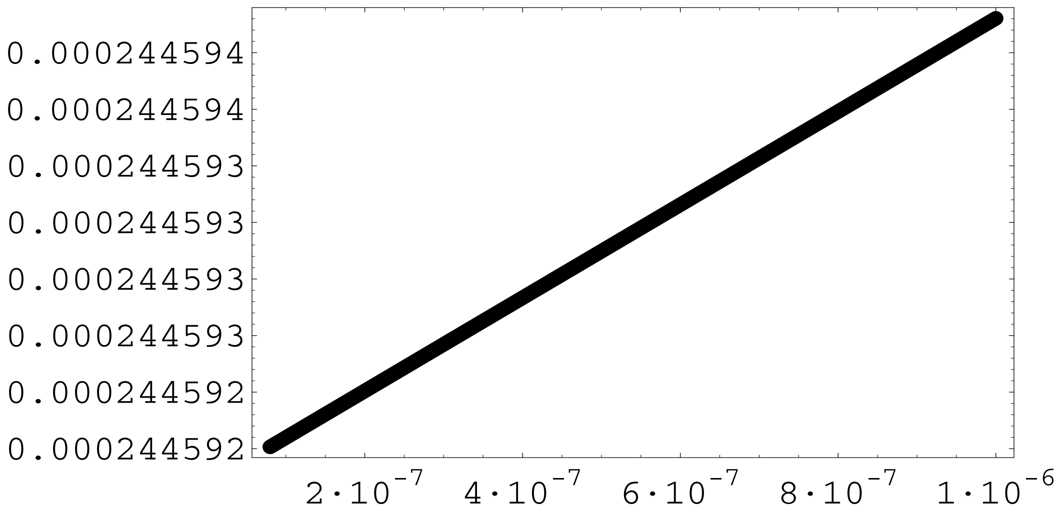


Figure 2: Square of Step Size vs Relative Error in v

PARAMETER ESTIMATION (INVERSE METHOD)

Problem: Given observed data

$$\begin{cases} \Pi = \int_z u(t, z)\, dz \\ \Theta = \int_z \int_x v(t, x, z)\, dx \, dz \end{cases}$$

Equation 4

Find parameter vector $q = (a, b, \lambda, e, \alpha, \gamma, \eta, \rho)$, where $q \in Q$ $(\infty - \dim \ space)$, so as to minimize

$$J(q) = \sum_t \left(\left| \int_z u(\cdot, q)\, dz - \Pi \right|^2 + \left| \int_z \int_x v(\cdot, q)\, dxdz - \Theta \right|^2 \right)$$

Equation 5

where $(u(q), v(q))$ is a parameter-dependent solution.

Step 1: Approximate solution via finite difference scheme
Step 2: Approximate Q via finite-dim compact $\{Q^M\}$

Step 1 and Step 2 give approximations $J_{\Delta t, \Delta x, \Delta z}(q)$ and $J_{\Delta t, \Delta x, \Delta z}(q^M)$

Fact: $J_{\Delta t,\Delta x,\Delta z}\left(q^{M}\right) \to J_{\Delta t,\Delta x,\Delta z}\left(q\right) \to J\left(q\right)$ as $\Delta t, \Delta x, \Delta z \to 0$ and $M \to \infty$ establishing existence of minimizer for $J\left(q\right)$

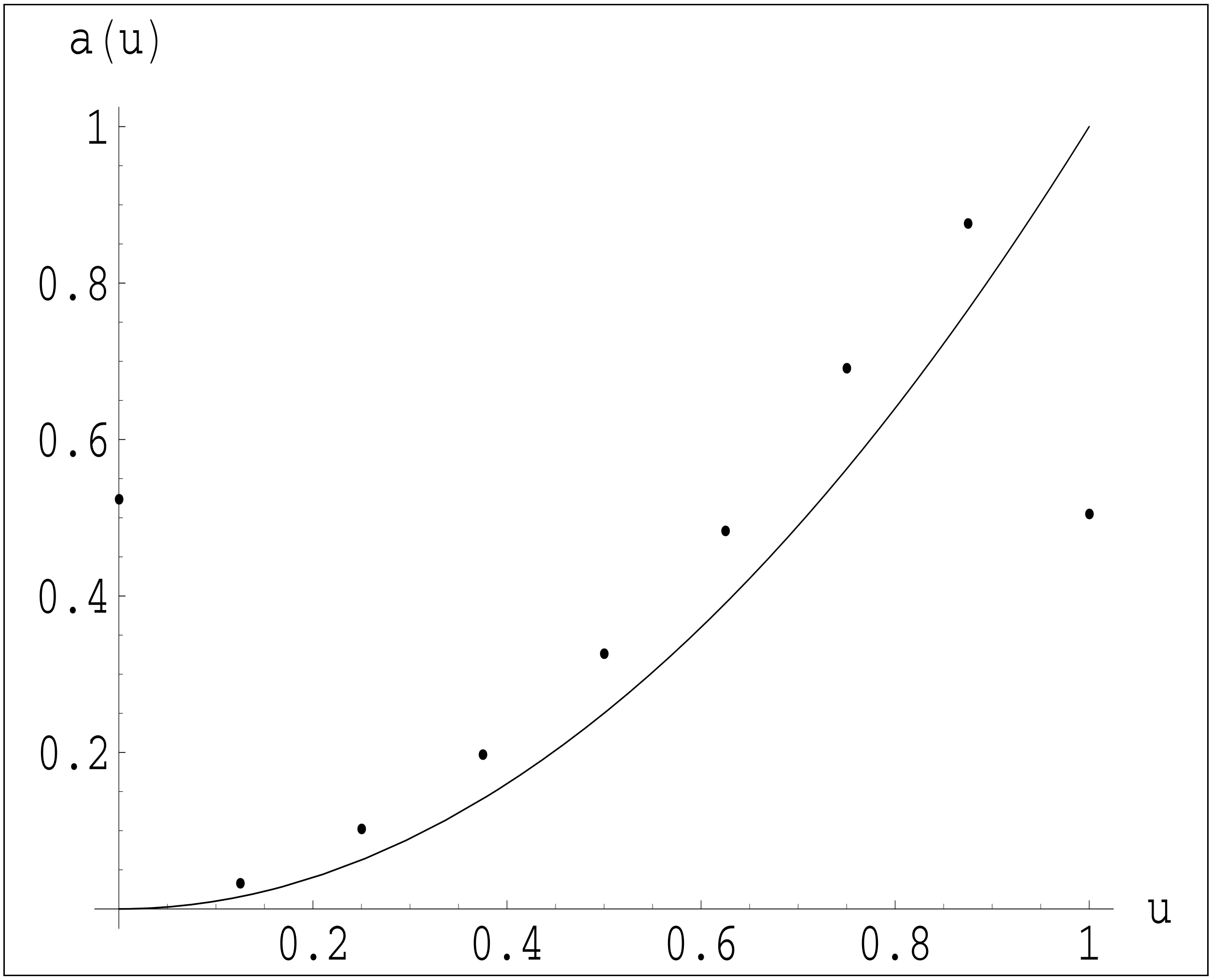


Figure 3: Exact vs Estimated a=u^2 with M=7

FUTURE RESEARCH

1. Apply Model Results to Experimental Data
2. Introduce use of Super-Computing thereby enabling applicability to real-world much larger domains in space and time

REFERENCES

[1] E. A. Sudicky & E. O. Frind, *Contaminant Transport in Fractured Porous Media: Analytic Solutions For a System of Parallel Fractures*, Water Resources Research **18(6)**, 1982.

[2] R. L. Drake & J. Chen, *Contaminant Transport in Parallel Fractured Media: Sudicky and Frind Revisited,* Submitted for Publication, 2003.

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